MATH 5A - TEST
Spring 2022
(Chapter 3.9, 4 & 5.1)

100 POINTS

NAME: ____________________

Show your work on this paper.

(1) Evaluate the following integrals. (3 points each)

(a) $\int \cos 6x \, dx$ $\quad u = 6x$, $du = 6dx$

$\frac{1}{6}\int \cos u \, du = \frac{1}{6}\sin u + C = \boxed{\frac{1}{6}\sin(6x) + C}$

remember the constant for all indefinite integrals

(b) $\int \frac{1}{x^3} dx = \int x^{-3} dx = \frac{x^{-2}}{-2} + C = \boxed{-\frac{1}{2x^2} + C}$

super easy to check indefinite integrals, just differentiate:

$\frac{d}{dx}\left(\frac{-1}{2x^2}\right) = \frac{d}{dx}\left(\frac{-1}{2}x^{-2}\right)$

$= x^{-3} = \frac{1}{x^3}$ ✓

(c) $\int 4\sec^2 z \, dz = 4\tan z + C$

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(2) Find the derivative of the function $g(x) = \int_2^{x^2} \sin^3 t \, dt$ (3 points)

This would be a function of x^2

Let $u = x^2$

$g'(x) = \frac{d}{dx}\int_2^{x^2} \sin^3 t \, dt = \frac{d}{dx}\int_2^{u} \sin^3 t \, dt$

$= \frac{d}{du}\left(\int_2^u \sin^3 t \, dt\right)\frac{du}{dt}$

$= \sin^3 u \frac{du}{dt}$

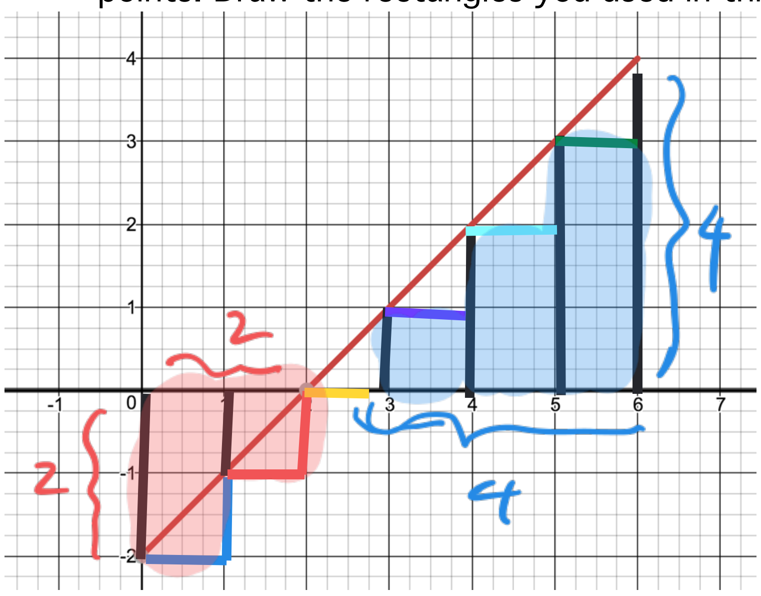
$= \sin^3(x^2)\, 2x$

can you actually explain what's going on here?

(3) In this problem you will evaluate $\int_0^6 (x-2)\,dx$ using the 4 methods discussed in class. (19 points)

a) Estimate the value of $\int_0^6 (x-2)\,dx$ using n= 3 subintervals and using the left endpoints as sample points. Draw the rectangles you used in this approximation.

$\Delta x = \frac{b-a}{n} = \frac{6-0}{6} = 1$



$1(f(0)+f(1)+f(2)+f(3)+f(4)+f(5))$

$-2+-1+0+1+2+3$

$\boxed{3}$

b) Find the exact value using the Riemann sum definition with sample points being right endpoints and the fact that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

$\Delta x = \frac{b-a}{n} = \frac{6}{n}$ $\quad x_i = a + i\Delta x = 0 + i\frac{6}{n}$

$f(x_i) = i\frac{6}{n} - 2$

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i6}{n} - 2\right)\left(\frac{6}{n}\right)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{36}{n^2} i - \frac{12}{n}\right)$$

$$= \lim_{n\to\infty} \frac{36}{n^2}\frac{n(n+1)}{2} - \frac{12}{n}\cdot n$$

$$= \lim_{n\to\infty} \frac{18(n+1)}{n} - 12 = \lim_{n\to\infty}\left(18 + \frac{18}{n} - 12\right) = \boxed{6}$$

c) Compute $\int_0^6 (x-2)\,dx$ using the area interpretation.

Area above − area below

$\frac{1}{2}\cdot 4\cdot 4 - \frac{1}{2}\cdot 2\cdot 2 = \boxed{6}$

They all match, as they should.

d) Compute $\int_0^6 (x-2)\,dx$ using the FTC part 2 and the antiderivative.

$\frac{1}{2}x^2 - 2x \Big]_0^6 = 18 - 12 = \boxed{6}$

(4) Evaluate the following integrals. Give simplified, exact answers. (7 points each)

(a) On this problem only, you MUST make a u-substitution and change to U's limits. On subsequent definite integrals you can choose to switch to u's limits or not, but use proper notation.

$\int_0^{\pi/2} \cos x \sqrt{\sin x}\ dx$

$u=\sin x$ $\quad$ $x=\frac{\pi}{2}$ $\quad$ $u=\sin x$

$du=\cos x\,dx$ $\quad$ $x=0$ $\quad$ $u=1$

$u=0$

$\int_0^1 u^{1/2}du = \frac{2}{3}u^{3/2}\Big]_0^1 = \boxed{\frac{2}{3}}$

(b) $\int \frac{\sqrt{t}-7t^2}{t^2}dt = \int\left(\frac{\sqrt{t}}{t^2}-\frac{7t^2}{t^2}\right)dt = \int(t^{-3/2}-7)dt$

$= -2t^{-1/2}-7t+C$

$= \boxed{\frac{-2}{\sqrt{t}}-7t+C}$

$u=1-3x$

(c) $\int_{2/3}^{3} \frac{1}{\sqrt[3]{1-3x}}\,dx$

$u=1-3x$ $\quad$ $x=3$ $\quad$ -8

$du=-3dx$ $\quad$ $x=\frac{2}{3}$ $\quad$ -1

$-\frac{1}{3}\int_{-1}^{-8}\frac{1}{\sqrt[3]{u}}du = -\frac{1}{3}\int_{-1}^{-8}(u^{-1/3})du = -\frac{1}{3}\frac{3}{2}u^{2/3}\Big]_{-1}^{-8}$

$= -\frac{1}{2}\left((-8)^{2/3}-(-1)^{2/3}\right) = -\frac{1}{2}(4-1) = \boxed{-\frac{3}{2}}$

(d) $\int_{-1}^{3}(5x-|x|)\,dx$

$|x| = \begin{cases} x & \text{if } x\geq 0 \\ -x & \text{if } x<0 \end{cases} \Rightarrow$ $5x-|x| = 4x$ if $x\geq 0$

$5x-|x| = 6x$ $\quad$ $x<0$

$\int_{-1}^{0} 6x\,dx + \int_0^3 4x\,dx$

$3x^2\Big]_{-1}^{0} + 2x^2\Big]_0^3$

$-3+18 = \boxed{15}$

(4 continued)

(e) $\displaystyle\int \frac{\cos\left(\frac{1}{x}\right)}{3x^2}\,dx$

$u = \frac{1}{x} = x^{-1}$

$du = -x^{-2}dx = -\frac{1}{x^2}dx$

$= -\frac{1}{3}\int \cos u\,du = -\frac{1}{3}\sin u + C = -\frac{1}{3}\sin\left(\frac{1}{x}\right) + C$

(f) $\displaystyle\int x^3\sqrt{x^2+1}\,dx$

$u = x^2+1$

$du = 2x\,dx$

$x^2 = u-1$

need x^2 in terms of u

$\frac{1}{2}\int x^2\, x\sqrt{x^2+1}\,dx$

$\frac{1}{2}\int (u-1)\sqrt{u}\,du = \frac{1}{2}\int \left(u^{3/2} - u^{1/2}\right)du$

$= \frac{1}{2}\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right) + C$

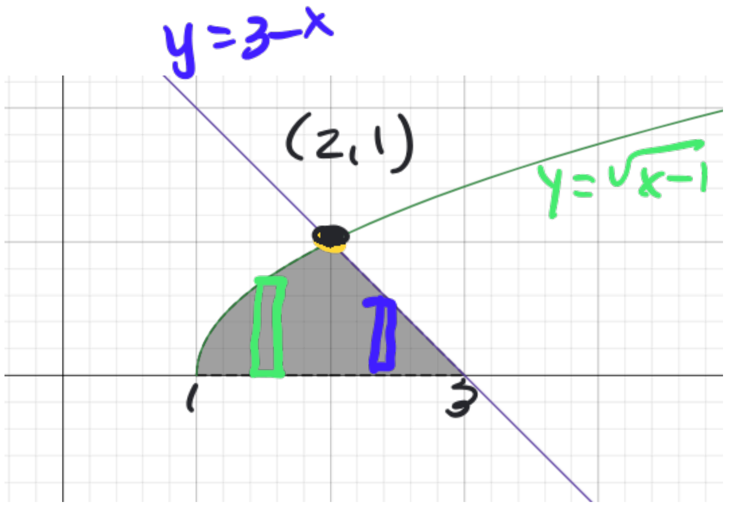
$= \boxed{\frac{1}{5}(x^2+1)^{5/2} - \frac{1}{3}(x^2+1)^{3/2} + C}$

(g) $\displaystyle\int_{-1}^{1} \frac{x}{\sqrt[3]{1+x^2}}\,dx$

short way, notice integrand is odd on $[-1,1]$ so integral is $\boxed{\text{zero}}$.

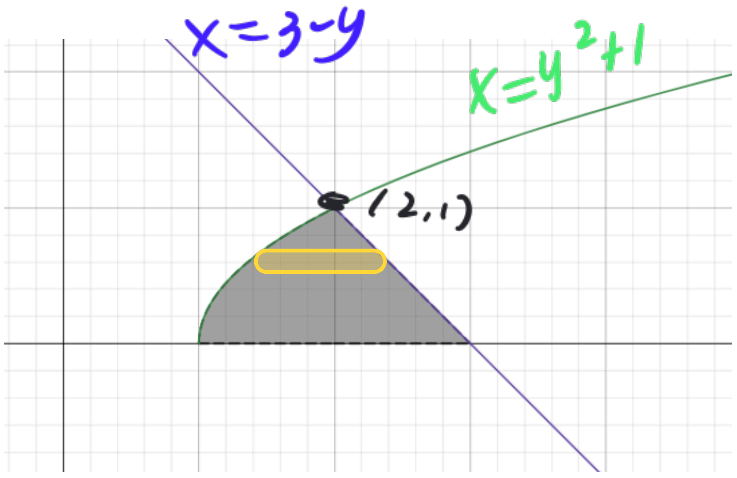
(6) Given the region bounded by the graphs of $y=\sqrt{x-1}$; $y=3-x$, and the x axis (12 points)

(a) Set up, but do not evaluate, an integral expression to find the area by integrating with respect to x.



$$\int_1^2 \sqrt{x-1}\,dx + \int_2^3 (3-x)\,dx$$

(b) Set up, but do not evaluate, an integral expression to find the area by integrating with respect to y.



$$\int_0^1 \left((3-y)-(y^2+1)\right)dy$$

(c) Find the area by evaluating one of the above. If time - do both ways to check

$$\int_0^1 (2-y-y^2)\,dy = 2y-\tfrac{1}{2}y^2-\tfrac{1}{3}y^3\Big]_0^1 = 2-\tfrac{1}{2}-\tfrac{1}{3} = \boxed{\tfrac{7}{6}}$$

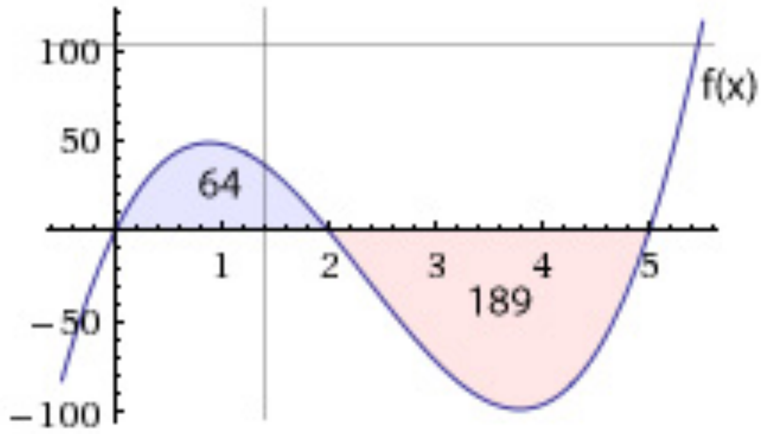
or

$$\int_1^2 \sqrt{x-1}\,dx + \int_2^3 (3-x)\,dx = \tfrac{2}{3}(x-1)^{3/2}\Big]_1^2 + 3x-\tfrac{1}{2}x^2\Big]_2^3$$

$u=x-1$

$$= \tfrac{2}{3} + \tfrac{9}{2} - 4 = \tfrac{31}{6} - \tfrac{24}{6} = \boxed{\tfrac{7}{6}}$$

(7) (8 points)



Aree above − Area Below
64 − 189
− 125

(a) Given the graph of y=f(x) and the areas shown in the figure above, find the following.

$\int_0^2 f(x)dx=$ 64 $\int_2^5 f(x)dx=$ −189 $\int_0^5 f(x)dx=$ −125

(b) Write an integral expression in terms of f(x) which would give the total enclosed area.

$$\int_0^2 f(x)dx - \int_2^5 f(x)dx$$